J. Fluid Mech. (1990), *vol.* 219, *pp.* 181–193
Printed in Great Britain

Breakage models: lognormality and intermittency

By HIDEKATSU YAMAZAKI
The Johns Hopkins University, Chesapeake Bay Institute, The Rotunda, 315, 711 West 40th Street, Baltimore, MA 21211, USA

(Received 25 September 1989 and in revised form 2 March 1990)

A breakage model for the statistical distribution of the dissipation rate is proposed: this model, B-model, is a modification of the Gurvich & Yaglom model (1967) taking the criticism of Mandelbrot (1974) into account. The B-model uses the beta distribution for the breakage coefficient α. The universal power spectrum of velocity for the B-model has a slightly flatter slope (positive correction) than the '$-\frac{5}{3}$' in contrast to all other previously proposed models, and this positive correction agrees with a theoretical argument made in Yakhot *et al.* (1989). The B-model predicts the structure functions of velocity observed by Anselmet *et al.* (1984) remarkably well without an empirical fit to the data.

1. Introduction

In his refinement of his original 1941 hypotheses Kolmogorov (1962) proposed a lognormal model for the probability density function (p.d.f.) of the dissipation rate, ϵ. Because the original hypotheses (Kolmogorov 1941*a*) were contended against Landau's argument in which the hypotheses overlooked the statistical characteristics of ϵ, the refinement was required. Gurvich & Yaglom (1967; hereinafter referred to as GY) then derived the theory of lognormality for ϵ extending Kolmogorov (1962). This theory is an application of breakage model (Kolmogorov 1941*b*) in the turbulence energy cascade. The lognormality of ϵ has been extensively studied by various researchers and seems successful as a first-order approximation model for the statistics of turbulence. Mandelbrot (1974), however, argued that the theory is not useful in estimating higher-order moments of turbulence statistics, and he proposed a fractal model for the self-similarity of turbulent cascade, which is analogous to the breakage model. Both models describe the statistics of turbulent field: p.d.f. of ϵ, the power spectrum and the structure function of turbulent velocity.

Since p.d.f. determines statistics of a random variable, the lognormality of ϵ is helpful because this p.d.f. requires only two parameters: the mean and the variance of $\log \epsilon$. Having estimated these parameters we can infer any statistics of ϵ. The distribution of ϵ also modifies Kolmogorov's original universal slope, the '$-\frac{5}{3}$', of turbulent velocity spectrum in the inertia subrange (Kolmogorov 1962). The GY lognormal model (hereinafter referred to as the GY model) predicts a negative correction; namely, the universal slope is steeper than the '$-\frac{5}{3}$'. All previously proposed models, including recent multifractal models, show a negative correction as well. Unfortunately, this deviation from the '$-\frac{5}{3}$' is presumably undetectable, no researcher has yet reported the departure of a power spectral slope from the '$-\frac{5}{3}$' slope. Recently Yakhot, She & Orszag (1989) argue that such a correction (negative) may be inconsistent with the dynamics of turbulence. They suggest that the correction to the '$-\frac{5}{3}$' should be a positive, namely a flatter slope than the '$-\frac{5}{3}$'. We see that a modified GY model presented in this work can predict a positive correction to the '$-\frac{5}{3}$'.

The velocity structure function is another way to study the statistical characteristics of turbulence. Since the turbulent velocity in the inertial subrange only depends on ϵ (Kolmogorov 1962), the structure function is directly related to the p.d.f. of ϵ, therefore, a proper model for the p.d.f. of ϵ should describe the statistics of turbulent velocity. Anselmet *et al.* (1984) report higher-order structure function of turbulent velocity up to 18th order, and they show that both the GY model and the β-model (a fractal model) are inconsistent with their observations. A random β-model of Benzi *et al.* (1984) – a multifractal model – improves the failure of the β-model *fitting* the observed higher-order moments. An attempt is made in this study to improve the GY model, and a new model is tested against the data in Anselmet *et al.* (1984).

The purpose of this study is to derive a new lognormal model taking the theoretical problem, pointed out by Mandelbrot (1974, 1976) for the GY model, into consideration. In order to identify this problem we briefly review the GY theory in the next section, then the new model (B-model) is derived. Both the GY model and the B-model are lognormal models and they differ, in principle, from a fractal model. Therefore, a brief summary of the fractal model is also presented. Section 3 compares the B-model to the GY-model and the fractal model. The conclusions of this study can be found in the last section.

2. Breakage models

2.1. *The Gurvich–Yaglom model*

In order to identify the theoretical deficiency in the GY model we briefly review this theory. The original idea of the breakage model is to explain the distribution of particles, which are successively crushed into several pieces at each breakage step (Shimizu & Crow 1988). Kolmogorov (1941*b*) showed the distribution is asymptotically lognormal after several breakages. GY, then, applied this idea to a turbulent energy cascade process to describe the statistical characteristics of ϵ. In principle the breakage model is based on a purely statistical consideration, the interpretation of this model is the significant contribution of GY.

Suppose the domain of an energy containing eddy size, L, is Q and is proportional to L^3. The domain average of ϵ is $\langle\epsilon\rangle$:

$$\langle\epsilon\rangle = Q^{-1}\int_Q \epsilon(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \tag{1}$$

where the local dissipation rate $\epsilon(\boldsymbol{x})$ is

$$\epsilon(\boldsymbol{x}) = 2\nu \sum (s(\boldsymbol{x})_{ij})^2, \tag{2}$$

and the strain rates are

$$s_{ij}(\boldsymbol{x}) = \frac{1}{2}\left(\frac{\partial u_i(\boldsymbol{x})}{\partial x_j}+\frac{\partial u_j(\boldsymbol{x})}{\partial x_i}\right) \quad \text{for } i,j = 1,\ 2 \text{ and } 3, \tag{3}$$

and both the turbulent velocity components, u_i, and the coordinate system, x_i, follow the conventional index notation, and $\boldsymbol{x}$ is a spatial vector (x_1, x_2, x_3). The original domain is successively divided into a subdomain, q_i, whose lengthscale is l_i, and the average of ϵ in a volume q_i is denoted as ϵ_i:

$$\epsilon_i = (q_i)^{-1} \int_{q_i} \epsilon(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{4}$$

The ϵ_i is a random variable representing the average of dissipation rate within q_i. The breakage coefficient α_i is defined as a ratio of two successive ϵ_i:

$$\alpha_i = \epsilon_i\,\epsilon_{i-1}^{-1} \quad \text{for } i = 1, \ldots, N_\mathrm{b}, \tag{5}$$

where N_b is the number of breakage processes. In the GY model, the ratio of lengthscales l_{i-1} and l_i for two successive breakages is a constant:

$$\lambda \equiv l_{i-1}\,l_i^{-1}, \tag{6}$$

thus, the breakages may not represent the actual breakdown of turbulent eddies. This is merely a geometrical decomposition of the spatial domain. We set $l_{N_\mathrm{b}} = r$ and $\epsilon_{N_\mathrm{b}} = \epsilon_r$ for the sake of consistency with other authors. Then at the N_bth breakage the volume average dissipation rate in a single cell, ϵ_r, can be expressed in terms of $\langle\epsilon\rangle$ by

$$\log \epsilon_r = \log\langle\epsilon\rangle + \sum_{i=1}^{N_\mathrm{b}} \log \alpha_i. \tag{7}$$

By virtue of Kolmogorov's (1962) third hypothesis the set of $\{\log\alpha_1, \ldots, \log\alpha_{N_\mathrm{b}}\}$ is mutually independent identically distributed random variables for velocity component in the inertial subrange. The GY model assumes that the random variable $\log\alpha_i$ follows a normal distribution. GY adopted the central-limit theorem to show that the random variable ϵ_r asymptotically distributes as lognormal; however, $\log\alpha_i$ is Gaussian according to GY's assumption so that *it is not necessary to apply the central-limit theorem to the theory* – a sum of normal distributions is always normal. Another implicit assumption being made in the GY model is that the probability of an $\alpha_i = 0$ event is zero, in other words, all of $N_\mathrm{c} = (Q/r^3)$ cells contain a dissipating region. As long as turbulent flows exist in the field, the dissipation of the kinetic energy should take place at a certain magnitude everywhere; therefore, this assumption seems realistic. On the other hand, the fractal model, which is summarized later, assumes that a certain fraction of the turbulent field is occupied by a non-dissipating region.

Applying the laws of probability, the mean m_r and variance σ_r^2 of $\log\epsilon_r$ are

$$m_r = \log\langle\epsilon\rangle + \xi \log_\lambda (Lr^{-1}), \tag{8}$$

$$\sigma_r^2 = \mu \log_\lambda (Lr^{-1}), \tag{9}$$

where $\log_\lambda (Lr^{-1})$ is the number of breakage processes N_b, and the universal constants ξ and μ are the mean and the variance of the random variable $\log\alpha$:

$$\xi = E[\log\alpha], \tag{10}$$

$$\mu = E[(\log\alpha - \xi)^2]. \tag{11}$$

Thus, μ is inherently positive. GY added spatial dependence terms in (8) and (9), which we ignore here because these terms are irrelevant to our discussion. They also approximated the number of breakages $\log_\lambda (Lr^{-1}) \approx \log_\mathrm{e} (Lr^{-1})$. This approximation requires either $\lambda \approx 3$ or the lengthscale r satisfying $\eta \ll r \ll L$. GY considers the latter case, as a result, their lognormality is strictly applicable in the inertial subrange of the velocity spectrum. Despite the fact that λ determines the number of breakage steps, which may relate to the actual breakdown of a large eddy to smaller eddies, less attention has been paid to this parameter. Most fractal models, for instance the β-model, assume $\lambda = 2$ (Frisch, Sulem & Nelkin 1978).

The geometry of the breakage process constrains the expected value of α to be unity as discussed in GY:

$$E[\alpha] = 1. \tag{12}$$

Since the GY model assumes a lognormal distribution for α, this constraint is automatically satisfied if $\xi = -\frac{1}{2}\mu$; consequently, the volume average dissipation rate does not depend on r. In fact GY assume $\xi = -\frac{1}{2}\mu$, and this assumption has let everyone use μ as the intermittency coefficient. The assumption of lognormality for α is convenient, but this assumption results in various inconsistencies as presented by Mandelbrot (1974). For instance, Mandelbrot states the moments of ϵ_r exist up to a finite order, and the structure function of the velocity is a convex function of the order. Mandelbrot (1976) states, 'one *must not*, and we *shall not*, replace $\log\alpha$ by its Gaussian approximation'.

2.2. *B-model*

Since the major problem in the GY model is the assumption of lognormality for α, we attempt to improve the GY model using an alternative distribution for α. Suppose as an extreme case the entire dissipation takes place in a single cell among all subdivided cells, the maximum value of α is $\alpha_{\max} \equiv \lambda^3$. This must be a physically constrained absolute maximum value because any singular points possibly generated from the Navier–Stokes equation should be smeared out by viscosity. Therefore, α must satisfy $0 < \alpha < \alpha_{\max}$, and the distribution must be defined in this finite domain. A possible distribution satisfying these conditions is the beta distribution (Mood, Graybill & Boes 1974), which is defined in a finite domain by three independent parameters, and can be applied to the variety of random variables. Hereinafter our model is called the B-model. Although no prior justification exists to use the beta distribution, at least this distribution satisfies the physical constraint for $\alpha_{\max}$; all other well-known continuous distributions are supported in an infinite domain. Recently Hosokawa (1989) applied a square-root exponential distribution for α arguing that velocity gradients and vorticity spatially distribute exponentially, but the domain of this distribution is also infinite.

A random variable α of the beta distribution supported in the range $(0, \alpha_{\max})$ has the following p.d.f.

$$f_\alpha(\alpha; a, b, \alpha_{\max}) = B(a,b)^{-1}\,\alpha_{\max}^{1-a-b}\,\alpha^{a-1}(\alpha_{\max}-\alpha)^{b-1} \quad \text{for } (0, \alpha_{\max}), \tag{13}$$

where a and b are positive parameters, and $B(a,b)$ is the beta function. As we stated earlier the geometrical constraint of the breakage process requires $E[\alpha] = 1$, thus the parameter b must be equal to $(\alpha_{\max}-1)\,a$. The universal constants ξ and μ for this distribution are

$$\begin{aligned}\xi &= E[\log\alpha] \\ &= \log\alpha_{\max} + \Psi(a) - \Psi((\alpha_{\max}-1)\,a),\end{aligned} \tag{14}$$

and

$$\begin{aligned}\mu &= E[(\log\alpha-\xi)^2] \\ &= \{\Psi(a)-\Psi(\alpha_{\max}\,a)\}^2 + \Psi'(a) - \Psi'(\alpha_{\max}\,a) - \{\Psi(a)-\Psi((\alpha_{\max}-1)\,a)\}^2,\end{aligned} \tag{15}$$

where $\Psi(x)$ is the psi function (Gradshteyn & Ryzhik 1980) and expressed in terms of the Gamma function $\Gamma(x)$:

$$\Psi(x) = \mathrm{d}\{\log \Gamma(x)\}/\mathrm{d}x, \tag{16}$$

and

$$\Psi'(x) = \mathrm{d}^2\{\log \Gamma(x)\}/\mathrm{d}x^2. \tag{17}$$

Therefore, ξ and μ can be equated with a parameter a for a given $\alpha_{\max} = \lambda^3$.

GY derived the structure function of volume average dissipation rate, $R_\epsilon(r)$, under the assumption that $\log\alpha$ is Gaussian. We abandon this assumption, but the final results (spectral slopes, etc.) are similar. For the sake of simplicity, we consider a one-dimensional case. The structure function, $R_\epsilon(r) = E[\epsilon_r(x)\,\epsilon_r(x+r)]$, can be equated to form the second-order moment of ϵ_r (Monin & Ozmidov 1985, p. 70) as follows:

$$M_2 = (2/r^2)\int_0^r\int_0^{r'} R_\epsilon(r'')\,\mathrm{d}r''\,\mathrm{d}r'. \tag{18}$$

Then,

$$\begin{aligned} R_\epsilon(r) &= \tfrac{1}{2}\mathrm{d}^2/\mathrm{d}r^2(r^2M_2) \\ &= \langle\epsilon\rangle^2(Lr^{-1})^{2\xi+2\mu} \\ &= \langle\epsilon\rangle^2(Lr^{-1})^{\theta}, \end{aligned} \tag{19}$$

where θ is the intermittency coefficient and is equal to $2\xi+2\mu$. Note that we assume the breakage process is applicable in the inertial subrange of velocity power spectrum, so we adopt the same approximation of $\log_\lambda(L/r) \approx \log_e(L/r)$ as done in the GY model. The corresponding dissipation spectrum $S_\epsilon(k)$ follows

$$S_\epsilon(k) \propto k^{-1+\theta}, \tag{20}$$

and this equation reduces to

$$S_\epsilon(k) \propto k^{-1+\mu} \tag{21}$$

for the GY model. All previous authors interpret μ as the intermittency coefficient instead of θ, and this difference is crucial for our discussion in this paper.

The GY model leads to the same universal spectral slope of turbulent velocity with Kolmogorov (1962). The B-model, however, changes the power dependence law slightly. The '$-\frac{5}{3}$' law becomes

$$E(k) = B\langle\epsilon\rangle^{\frac{2}{3}}k^{-\frac{5}{3}+\frac{2}{3}(\xi+\mu/3)} \tag{22}$$

where B is a universal constant. The nth order structure function of velocity is

$$\begin{aligned} R_n(\boldsymbol{r}) &= \langle(\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r})-\boldsymbol{u}(\boldsymbol{x}))^n\rangle \\ &= A_n\langle\epsilon_r^{n/3}\rangle\, r^{n/3} \\ &= A_n\langle\epsilon\rangle^{n/3}r^{n/3}(Lr^{-1})^{n\xi/3+n^2\mu/18}, \end{aligned} \tag{23}$$

where A_n is a constant.

2.3. *Fractal models*

Recent developments in the area of fractals (Mandelbrot 1982) shed light on the intermittency of turbulence (Mandelbrot 1976; Procaccia 1984; Turcotte 1988). A fractal is a self-similar process model and is tightly related to breakage models. In order to compare the fractal model to the GY model and the B-model, we briefly summarize these models.

Suppose a cubic domain L^3 is divided into N_c number of r^3 cells, and the dissipation is taking place only in N_f cells. The fractal dimension D is defined as follows:

$$N_f = (Lr^{-1})^D. \tag{24}$$

The average of ϵ in a single cell, $\{\epsilon_r\}$, can be expressed as:

$$\{\epsilon_r\} = \langle\epsilon\rangle N_c N_t^{-1} = \langle\epsilon\rangle (Lr^{-1})^{3-D}, \tag{25}$$

where $\{\cdot\}$ denotes an average over non-empty cells. An average over all available cells, $\langle\epsilon_r\rangle$, is unchanged from $\langle\epsilon\rangle$; thus, the equivalent definition of breakage coefficient, α, is always one. The probability of finding non-empty cells is $(Lr^{-1})^{D-3}$, therefore, $D = 3$ means the probability is one. Mandelbrot (1976) argues that D must be larger than two for a topological reason. The fractal dimension D for a fully developed turbulence cannot exceed three from a theoretical point of view (Foias, Manley & Teman 1987), so $2 < D < 3$, which is rather a wide range. Hentschel & Procaccia (1983) suggest $2.5 < D < 2.75$. Benzi & Vulpiani (1980) estimated $D \approx 3-\frac{2}{3}$ from the β-model (Frisch *et al.* 1978). A recent work by Benzi *et al.* (1984) suggests $D = 2.91$ from a modified β-model. Sreenivasan & Meneveau (1986) state $D = 2.6$.

The fractal dimension, D, is related to the intermittency coefficient, θ (Benzi *et al.* 1984):

$$D = 3-\theta. \tag{26}$$

Thus, there is an interconnection between the lognormal model and the fractal model. But despite the fact that the expression of $\langle\epsilon_r\rangle$ for the lognormal model, equation (7), and $\{\epsilon_r\}$ for the fractal model, equation (25), are similar the basic idea behind them is quite different. For the lognormal model we came up with a continuous probability density function, lognormal distribution; on the other hand, we deal with a discrete probability law for the fractal model. We also assume that a certain finite value of ϵ exists everywhere in the domain for the lognormal model, but for the fractal model there are $(Lr^{-1})^3-(Lr^{-1})^D$ numbers of cells which are non-dissipating regions. Frisch *et al.* (1978) note that the β-model is based on nonlinear energy transfer rather than a dynamically irrelevant local dissipation rate, such as the case of the lognormal models. On the other hand, Levich (1987) mentions that the fluctuations of nonlinear transfer terms are similar to the fluctuation of energy dissipation rate. Values of nonlinear energy transformation at a certain time may not be similar to local dissipation rate at that time, for instance vorticity and dissipation rate do not necessarily agree, but it seems reasonable to assume the nonlinear transfer and the dissipation are statistically identical.

An application of the idea of fractal led Frisch *et al.* (1978) to derive the β-model by incorporating the dynamics of turbulent energy cascade. They apply the breakage process in wavenumber domain assuming the turbulent energy cascade takes place locally in an adjacent wavenumber. Frisch *et al.* (1978) introduced the coefficient $\beta = \lambda^{D-3}$ in the energy cascade:

$$\epsilon_{i-1} = \beta\epsilon_i \tag{27}$$

where $\lambda = l_{i-1}/l_i$ is set to 2 in the β-model. They assume that a fraction β of inertia energy can be transferred from a wavenumber l_{i-1}^{-1} to an adjacent wavenumber l_i^{-1}. According to this model the decomposition of the original space is rather deterministic and dissipating regions become less space filling as the lengthscale reduces (Paladin & Vulpiani 1987).

Because the β-model has been compared to other models frequently, we also compare the β-model to the B-model derived in this study. The inconsistency in the higher-order moments exists in most intermittency models; this may well be incidental, so we should not take such an unfortunate nature too seriously as a

	B-model	GY-model	β-model
θ	$2\xi+2\mu$	μ	$\frac{2}{3}\mu\dagger$
χ	$\frac{2}{3}\xi+\frac{2}{9}\mu$	$-\frac{1}{9}\mu$	$-\frac{2}{9}\mu\dagger$
ζ_n	$\frac{1}{3}n-\frac{1}{3}n\xi-\frac{1}{18}n^2\mu$	$\frac{1}{3}n-\frac{1}{18}\mu n(n-3)$	$\frac{1}{3}n-\frac{2}{9}\mu(n-3)\dagger$

† Interpretation of μ is not the same with previous reports thus $\frac{2}{3}\mu$ should be read as 'conventional' μ.

TABLE 1. List of power law exponents for each model

reviewer suggested. But without improving the existing models we may not be able to gain a further insight into the structure of turbulence.

An improvement for the β-model has been attempted by generalizing the coefficient β. Benzi *et al.* (1984) introduced a random β-model allowing β to be a random variable. Their extension is a multifractal set which has recently become popular. Benzi *et al.* (1984) propose an empirical discrete probability density function for β:

$$f_\beta(\beta;c) = c\delta(\beta-0.5)+(1-c)\,\delta(\beta-1), \tag{28}$$

where c is a parameter. This idea is the exact analogy to the probability density function of α for the lognormal model.

3.4. *Summary of models*

Since we have presented three breakage models, one of them being the fractal model, we briefly summarize these models: (*a*) the GY model, (*b*) the B-model, (*c*) the β-model.

(*a*) The power law dependence on the structure function of dissipation is

$$R_\epsilon(r) \equiv E[\epsilon_r(x)\,\epsilon_r(x+r)] \propto r^{-\theta}, \tag{29}$$

and the corresponding dissipation spectrum is

$$S_\epsilon(k) \propto k^{-1+\theta}. \tag{30}$$

(*b*) The power law dependence on the velocity spectrum is

$$E(k) \propto k^{-\frac{5}{3}+\chi}. \tag{31}$$

(*c*) The nth order structure function of velocity is

$$R_n(r) \propto \langle\epsilon\rangle^{n/3}\,r^{-\zeta_n}. \tag{32}$$

Table 1 shows the power law exponents for each model.

3. Discussions

In this section we compare the B-model to the GY model and the β-model. The intermittency coefficient, θ, can be estimated from either (29) or (30) and takes the major role of the breakage models. The reported values of θ (table 2) vary between 0.2 and 0.6. Levich (1987) notes, 'Usually, it has been reported as close to $0.4 \sim 0.5$. Recently, it has been approximately settled as $0.15 < \theta < 0.25$, i.e. as much smaller.' The purpose of this study is not to estimate θ, so we will assume that 0.2 is the most plausible value. Anselmet *et al.* (1984), however, suggest that θ can also be close to 0.25, so we consider this value as the secondary candidate.

Source	θ	Description
Pond & Stewart (1965)	0.4	ABL†
Gurvich & Yaglom (1967)	0.4	ABL
Van Atta & Chen (1970)	0.5	ABL
Stewart *et al.* (1970)	0.35	ABL
Belyaev *et al.* (1975)	0.56 ± 0.11	Oceanic measurement (36–140 m)
Van Atta & Yeh (1975)	0.22	ABL
Fujisaka & Mori (1979)	0.34*	Theory using β-model
Benzi & Vulpiani (1980)	0.67	Theory
Van Atta & Antonia (1980)	0.25	Laboratory experiment & ABL
Nelkin (1981)	0.25	Theory
Antonia *et al.* (1981)	0.2	Laboratory experiment
Hentschel & Procaccia (1983)	0.25 − 0.5*	Fractal dimension
Anselmet *et al.* (1984)	0.2 ± 0.05	Laboratory experiment
Benzi *et al.* (1984)	0.09*	Fractal dimension of modified β-model
Sreenivasan & Meneveau (1986)	0.4*	Composite data

† Atmospheric boundary layer.

TABLE 2. List of reported intermittency coefficients θ. The value with an asterisk (*) is converted from the fractal dimension using equation (2*b*). Most direct measurements of θ appear between 0.2 and 0.5. The converted values of θ from fractal dimension have a much wider range. This suggests the conversion formula may not be correct

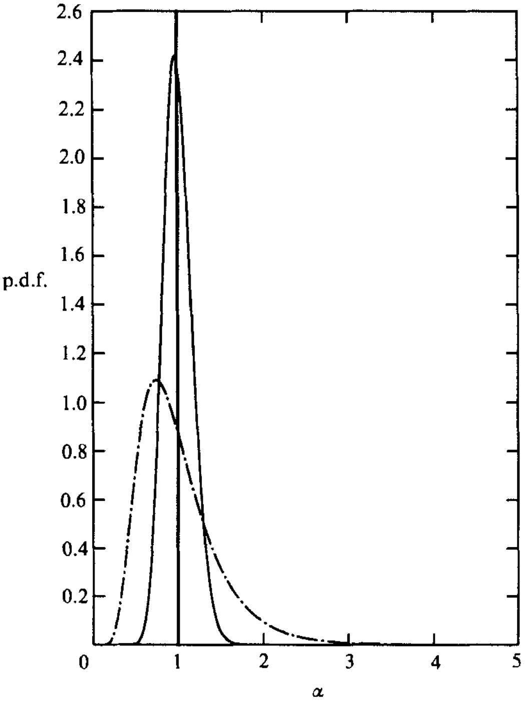

FIGURE 1. A comparison of lognormal distribution (chain-dot line) and beta distribution (solid line) for α. The scale ratio λ is 5 and θ is 0.2. The vertical thick line is α for the β-model which is a δ function.

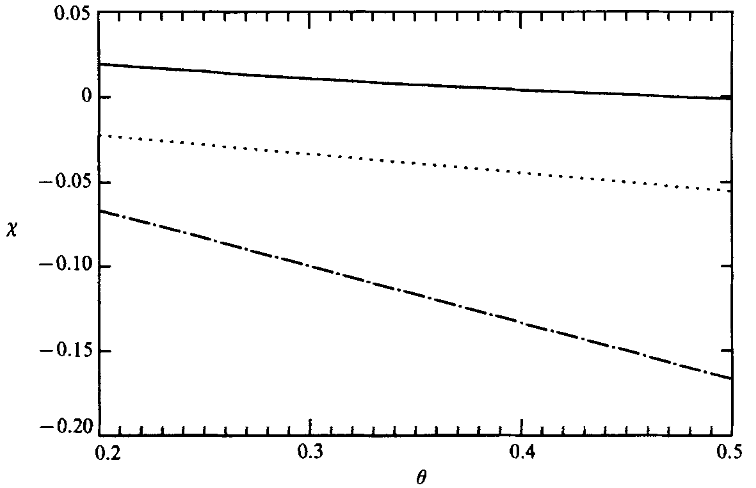

FIGURE 2. Correction factor χ for the universal spectrum slope. ——, B-model; ..., GY-model; —·—, β-model.

3.1. *Breakage coefficient*: α

In order to give an intuitive idea of how a beta distribution differs from a lognormal distribution we examine both distributions for a case ($\theta = 0.2$ and $\lambda = 5$; the Appendix provides the reason for $\lambda = 5$) in figure 1. Both distributions have almost negligible probability for $\alpha > 3$, and most realizations of α occur between 0 and 2. The lognormal distribution has a longer tail than that of the beta distribution, but the peak of the p.d.f. is about half of the beta distribution. The mode of the beta distribution occurs at $\alpha = 0.96$. The p.d.f. of α for the β-model is a delta function, namely $\delta(\alpha - 1)$ which is a deterministic value.

A physical interpretation of α is that the energy transformation of one eddy to the next generation is saturated in the wavenumber space at a constant rate $\langle \epsilon \rangle$ according to the Kolmogorov 1941 model and the β-model. This transformation of energy takes place locally in the wavenumber space, which may not be a correct assumption considering the highly nonlinear nature of turbulence (Levich 1987). On the other hand, the B-model and the GY model consider that the saturation is maintained at a probabilistic manner satisfying $E[\alpha] = 1$.

3.2. *Correction factor for universal spectrum*: χ

The value of χ (see equation (31)) for the B-model is positive in contrast to both the GY model and the β-model which predict a negative correction (figure 2). The GY model, however, has only 3% correction at the maximum $\theta = 0.5$, on the other hand, the β-model suggests 10% correction in the slope, if so, the departure from the '$-\frac{5}{3}$' slope may be detectable from experimental data (Frisch *et al.* 1978).

What is physically meant by a steeper or a flatter slope than the '$-\frac{5}{3}$' universal spectrum? Yakhot *et al.* (1989) mention that the β-model has a steeper slope than the '$-\frac{5}{3}$' because turbulence becomes more and more concentrated at smaller scales. All previously proposed intermittency models, including multifractal models, predict that the universal spectrum is steeper than the '$-\frac{5}{3}$'. Yakhot *et al.* (1989) propose the opposite possibility from a dynamical renormalization group analysis, namely the energy spectrum is flatter than the '$-\frac{5}{3}$'. They assume that energy transformation can take place in both local and nonlocal fashion from a large scale to smaller scales.

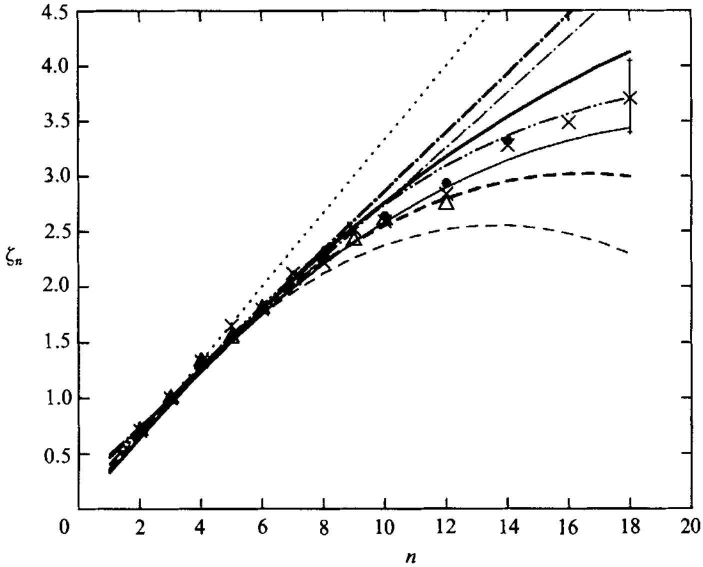

FIGURE 3. Power exponents of nth-order structure function. ...., Kolmogorov 1941 (the '$-\frac{5}{3}$' slope); —·—, β-model; —··—, random β-model; ——, B-model; ---, GY-model. Two values of θ are used; $\theta = 0.2$ (thick lines) and $\theta = 0.25$ (thin lines) which is the upper bound of estimated θ in Anselmet *et al.* (1984). Experimental values are compiled from Anselmet *et al.* (1984): ●, $Re = 3.3 \times 10^4$; △, $Re = 3.5 \times 10^4$; ×, $Re = 9.1 \times 10^4$. A vertical bar associated with the 18th order of × is the error bond shown in Anselmet *et al.* (1984). The Reynolds number, Re, is converted from their Taylor scale base Reynolds number, Re_λ using a formula given in Levich (1987), namely $Re = \frac{1}{8}Re_\lambda^2$.

It should be noted that the B-model is the only model, at least known to the author, predicting a flatter slope than the '$-\frac{5}{3}$', but the B-model does not require a nonlocal energy transformation. This difference should deserve attention in a future study.

3.3. *Power law coefficient for velocity structure functions*: ζ_n

Experimental values of ζ_n (see equation (32)) are compiled from Anselmet *et al.* (1984) to examine the model dependence of ζ_n. Exponents for higher-order statistics up to $n = 8$ show little difference among three models (figure 3). For $n > 12$, experimental values of Anselmet *et al.* (1984) exceed the GY model. The B-model, however, predicts the experimental values remarkably well, and these experimental values are between $\theta = 0.2$ and $\theta = 0.25$ for the B-model. On the other hand, the β-model exhibits a larger discrepancy from the experimental values for a high-order n. It should be noted that because ζ_n for the GY-model is a convex function with n, it holds a pathological characteristic as discussed in Mandelbrot (1974), namely ζ_n eventually becomes negative. The B-model holds the same tendency, but it is much slower than the GY-model. Although both lognormal models have the pathology in high orders, the fractal model has a physically inconsistent nature at zeroth order for the structure function, which shows an r dependence, namely $\zeta_0 = \theta$. This contradicts the mathematical definition of the structure function. To date no model has been proposed to satisfy the structure function for all orders.

Since the B-model is an improved version of the GY-model, it is fair to compare the counterpart of the β-model. According to the random β-model (multifractal) the power law coefficient ζ_n is expressed as follows (Benzi *et al.* 1984):

$$\zeta_n = \tfrac{1}{3}n - \log_2 E[\beta^{1-\frac{1}{3}n}], \tag{33}$$

where β distributes as the p.d.f. given in equation (28). Because this p.d.f. requires one unknown parameter, c, Benzi *et al.* (1984) fit the data of Anselmet *et al.* (1984) to obtain c. The fitted equation is shown in figure 3 (double dot dash). This artifact introduces a different value of the fractal dimension from the conventional relation, $D = 3-\theta$. Despite the fact that ζ_n of the random β-model is subjective, the asymptotic value of ζ_n is interesting because it converges to roughly four. Therefore, higher-order moments of turbulent velocity are more realistic than the lognormal models, the GY and the B-model. Recently Wong (1989) proposed a multifractal model extending Foias *et al.* (1987) and without an empirical fit his model shows a remarkable agreement of ζ_n with the data of Anselmet *et al.* (1984). Unfortunately Wong (1989) is a preliminary report so the detail is not available to us.

It may be instructive to consider a possible improvement of the lognormal models for the undesirable nature of higher-order structure function. The ζ_n of the lognormal models is a convex function of n, this is because the p.d.f. of ϵ_r has a longer tail than realistically possible high ϵ_r. Although the B-model restricts the maximum of α, the maximum value of ϵ_r is still infinity. In reality no matter how strong turbulence is, the ϵ_r must be bounded by a finite domain. Therefore, even if the central-limit theorem suggests that ϵ_r follows a lognormal distribution, we must terminate the long tail of this distribution depending on the strength of turbulence, so the maximum ϵ_r should be a Reynolds number dependent variable. Paladin & Vulpiani (1987) also introduce a Reynolds-number dependence in their multifractal model. A future study is called for this extension.

4. Conclusions

We modified the GY model in order to improve the lognormal model for the p.d.f. of the dissipation rate. As Mandelbrot (1974) identified the problem of the GY model, the crucial deficiency of this model is to assume a lognormal distribution for the breakage coefficient α. We applied the beta distribution for this coefficient and found that the new model, the B-model, explains experimental data of turbulence well. The major modifications are the following:

(i) The B-model predicts that the universal velocity spectral slope is flatter than the '$-\frac{5}{3}$'. This agrees with a recent theoretical result of Yakhot *et al.* (1989).

(ii) Higher-order structure functions, $n > 8$, observed from a laboratory experiment (Anselmet *et al.* 1984) follow the B-model well without an empirical fit as done in the random β-model.

I am grateful to T. R. Osborn for his encouragement of this study, and I benefitted from discussions with R. G. Lueck and O. M. Phillips. I am indebted to comments provided by two anonymous reviewers. The editorial assistance for this manuscript was provided by L. Wiggins. This work was supported by the Office of Naval Research.

Appendix. Numerical method to calculate parameters of the *B*-model

The calculations of (14) and (15) are rather cumbersome for the arbitrary choice of a and λ, but the computation can be considerably reduced if we assume the parameters to be integers. Then (16) and (17) can be expressed in terms of a finite sum of series (Gradshteyn & Ryzhik 1980). The statistics drawn from this distribution are the smooth function of integer a and λ, so that a proper interpolation method gives any combination of real values of a and λ.

As we mentioned in §3 the values of θ are between 0.2 and 0.5, so we are only interested in this range. There is no firm foundation for choosing λ. Here we assume λ as a prime number, because any non-prime number can be generated from a combination of prime numbers. For a given smallest size of cell, the number of offspring 'eddies', λ^3, determines the number of breakage processes, and the central-limit theorem requires the number of processes to be high. As a result, we hope to find a low prime number λ from the beta distribution. We found that 5 is the lowest prime number which gives the minimum θ as less than 0.2.

REFERENCES

ANSELMET, F., GAGNE, Y., HOPFINGER, E. J. & ANTONIA, R. A. 1984 High-order velocity structure functions in turbulent shear flows. *J. Fluid Mech.* **140**, 63–89.

ANTONIA, R. A., PHAN-THIEN, N. & SATYAPRAKASH, B. R. 1981 Autocorrelation and spectrum of dissipation fluctuations in a turbulent jet. *Phys. Fluids* **24**, 554–555.

BELYAEV, V. S., LUBIMTZEV, M. M. & OZMIDOV, R. V. 1975 The rate of dissipation of turbulence energy in the upper layer of the ocean. *J. Phys. Oceanogr.* **5**, 499–505.

BENZI, R., PALADIN, G., PARISI, G. & VULPIANI, A. 1984 On the multifractal nature of fully developed turbulence and chaotic system. *J. Phys.* A: *Math. Gen.* **17**, 3521–3531.

BENZI, R. & VULPIANI, A. 1980 Small-scale intermittency of turbulent flows. *J. Phys.* A: *Math. Gen.* **13**, 3319–3324.

FOIAS, C., MANLEY, O. P. & TEMAN, R. 1987 Self-similar invariant families of turbulent flows. *Phys. Fluids* **30**, 2007–2020.

FRISCH, U., SULEM, P. & NELKIN, M. 1978 A simple dynamical model of intermittent fully developed turbulence. *J. Fluid Mech.* **87**, 719–736.

FUJISAKA, H. & MORI, H. 1979 A maximum principle for determining the intermittency exponent μ of fully developed steady turbulence. *Prog. Theor. Phys.* **62**, 54–60.

GRADSHTEYN, I. S. & RYZHIK, I. M. 1980 *Table of integrals, series, and products*. Academic. 1160 pp.

GURVICH, A. S. & YAGLOM, A. M. 1967 Breakdown of eddies and probability distributions for small scale turbulence. *Phys. Fluids* **10**, 59–65.

HENTSCHEL, H. G. E. & PROCACCIA, I. 1983 Fractal nature of turbulence as manifested in turbulent diffusion. *Phys. Rev.* A **27**, 1266–1269.

HOSOKAWA, I. 1989 An advanced model of dissipation cascade in locally isotropic turbulence. *Phys. Fluids* A **1**, 186–189.

KOLMOGOROV, A. N. 1941*a* Local turbulent structure in incompressible fluids at very high Reynolds number. *Dokl. Akad. Nauk SSSR* **30**, 299–303.

KOLMOGOROV, A. N. 1941*b* On the logarithmical normal particle size distribution caused by particle crushing. *Dokl. Akad. Nauk SSSR* **31**, 99–102.

KOLMOGOROV, A. N. 1962 A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number. *J. Fluid Mech.* **13**, 82–85.

LEVICH, E. 1987 Certain problems in the theory of developed hydrodynamical turbulence. *Phys. Rep.* **151**, 129–238.

MANDELBROT, B. B. 1974 Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier. *J. Fluid Mech.* **62**, 331–358.

MANDELBROT, B. B. 1976 Intermittent turbulence and fractal dimension: kurtosis and the spectral exponent $\frac{5}{3}+\beta$. In *Turbulence and Navier–Stokes Equations* (ed. R. Teman). Lecture Notes in Mathematics, vol. 565, pp. 121–145. Springer.

MANDELBROT, B. B. 1982 *The Fractal Geometry of Nature*. Freeman.

MONIN, A. S. & OZMIDOV, R. V. 1985 *Turbulence in the Ocean*. Dordrecht: D. Reidel. 247 pp.

MOOD, A. M., GRAYBILL, F. A. & BOES, D. C. 1974 *Introduction to the Theory of Statistics*, 3rd edn. McGraw-Hill. 564 pp.

NELKIN, M. 1981 Do the dissipation fluctuations in high Reynolds number turbulence define a universal exponent? *Phys. Fluids* **24**, 556–557.

PALADIN, G. & VULPIANI, A. 1987 Anomalous scaling laws in multifractal objects. *Phys. Rep.* **154**, 147–225.

POND, S. & STEWART, R. W. 1965 Measurement of statistical characteristics of small-scale turbulence. *Izv. Akad. Nauk SSSR Fiz Atmosfi Okeana* **1**, 914–919.

PROCACCIA, I. 1984 Fractal structures in turbulence. *J. Stat. Phys.* **36**, 649–663.

SHIMIZU, K. & CROW, E. L. 1988 History, genesis, and properties. In *Lognormal Distributions: Theory and Applications* (ed. E. L. Crow & K. Shimizu). Marcel Dekker.

SREENIVASAN, K. R. & MENEVEAU, C. 1986 The fractal facets of turbulence. *J. Fluid Mech.* **173**, 357–386.

STEWART, R. W., WILSON, J. R. & BURLING, R. W. 1970 Some statistical properties of small scale turbulence in an atmospheric boundary layer. *J. Fluid Mech.* **41**, 141–152.

TURCOTTE, D. L. 1988 Fractals in Fluid Mechanics. *Ann. Rev. Fluid Mech.* **20**, 5–16.

VAN ATTA, C. W. & ANTONIA, R. A. 1980 Reynolds number dependence of skewness and flatness factors of turbulent velocity derivatives. *Phys. Fluids* **23**, 252–257.

VAN ATTA, C. W. & CHEN, W. Y. 1970 Structure function of turbulence in the atmospheric boundary layer over the ocean. *J. Fluid Mech.* **44**, 145–159.

VAN ATTA, C. W. & YEH, T. T. 1975 Evidence for scale similarity of internal intermittency in turbulent flows at large Reynolds numbers. *J. Fluid Mech.* **71**, 417–440.

WONG, V. C. 1989 On the similarity of the small-scale structure of turbulence. *Seventh Symposium on Turbulent Shear Flows, Stanford University, August 21–23, 1989*, pp. 30.4.1–30.4.6.

YAKHOT, V., SHE, Z. S. & ORSZAG, S. A. 1989 Deviations from the classical Kolmogorov theory of the inertial range of homogenous turbulence. *Phys. Fluids* A **1**, 289–293.